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A two-dimensional analogue of the Virasoro algebra

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Abstract

In this article, we study the cohomology of Lie algebras of vector fields of holomorphic type $Vect_{1,0}(M)$ on a complex manifold M. The main result is the introduction of a kind of order filtration on the continuous cochains on $Vect_{1,0}(M)$ and the calculation of the second term of the resulting spectral sequence. The filtration is very much in the spirit of the classical order filtration of Gelfand and Fuks, but we restrict ourselves to z-jets only for a local holomorphic coordinate z. This permits us to calculate the diagonal cohomology (because of the collapse of our spectral sequence) of $Vect_{0,1}(\Sigma)$ for a compact Riemann surface Σ of genus $g > 0$.

In the second section, we calculate the first three cohomology spaces of the Lie algebra $W_1 \otimes \mathbb{C}[[t]]$ which is regarded as the formal version of $Vect_{1,0}(\Sigma)$. In the last section, we recall why $Vect_{0,1}(\Sigma)$ can be regarded as the two-dimensional analogue of the Witt algebra. Then, we define, following Etingof and Frenkel, a central extension which is consequently a two-dimensional analogue of the Virasoro algebra — our cohomology calculations showing that it is a universal central extension.

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1. Introduction

The continuous cohomology of Lie algebras of $\mathcal{C}^{\infty}$-vector fields [1,6–8] has proven to be a subject of great geometrical interest: One of its most famous applications is the construction of the Virasoro algebra as the universal central extension of the Lie algebra of vector fields on the circle.

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Because of its interest in conformal field theory, it is tempting to generalize the Virasoro algebra to higher dimensions. In this perspective, the results of Gelfand and Fuks on the cohomology of W_n, the Lie algebra of formal vector fields in n variables, and the cohomology of $Vect(M)$, the Lie algebra of C^∞ vector fields on a manifold M, are disappointing, because they show on the one hand

$$H^p(W_n) = 0 \quad \text{for } 0 < p < 2n + 1$$

and on the other that there is a spectral sequence abuting to $H^*_\Delta(Vect(M))$, i.e. the diagonal cohomology of $Vect(M)$, with second term

$$E_2^{p,q} = H_{-p}(M) \otimes H^q(W_n).$$

These two results show that there cannot be a non-trivial central extension of $Vect(M)$ in case $\dim(M) > 1$.

Recall that the Virasoro algebra is characterized as being the universal central extension of the Lie algebra of outer derivations of the central extension of the Loop- or Kac–Moody algebra $C^\infty(S^1, Lie(G))$, G being a finite-dimensional, compact, simple, simply connected, connected Lie group with Lie algebra $Lie(G)$.

Now, Etingof and Frenkel [2] — along with Khesin, see also [3,5] — investigated the current group $C^\infty(\Sigma, G)$ as a promising two-dimensional analogue of the Loop group $C^\infty(S^1, G)$, see also [3,5]. They show ([2, Proposition 1.3]) that the Lie algebra of outer derivations of a g-dimensional central extension of the current algebra $C^\infty(\Sigma, Lie(G))$ is $Vect_{0,1}(\Sigma)$ for genus $g \geq 2$, and $Vect_{0,1}(\Sigma) \times \langle \partial/\partial z \rangle$ for $g = 1$.

Consequently, the universal central extension of $Vect_{0,1}(\Sigma)$ (resp. $Vect_{0,1}(\Sigma) \times (\partial/\partial z)$ for $g = 1$) — if it exists — is a promising analogue in two dimensions of the Virasoro algebra, leading possibly to an interesting representation theory (Sugawara construction in two dimensions, etc.).

In this article, we show that there exists a universal central extension of $Vect_{0,1}(\Sigma)$ with g-dimensional center, as expected. In order to do this, we investigate in the first section the cohomology of the formal version of $Vect_{0,1}(\Sigma)$, i.e. of $W_1 \otimes \mathbb{C}[[t]]$. It turns out to be the same as the one of W_1, at least in degrees up to 3. In Section 2, we “globalize” the result by means of the Gelfand–Fuks spectral sequence for diagonal cohomology, showing finally that not only the diagonal, but also the continuous cohomology of $Vect_{0,1}(\Sigma)$ is 0 in degree 1 and g-dimensional in degree 2. Section 3 recalls some known facts helping to construct finally the central extension.

2. Cohomology of $W_1 \otimes \mathbb{C}[[t]]$

Let W_1 be the Lie algebra of vector fields on the real line with complex coefficients. Define the usual Lie bracket on the tensor product $W_1 \otimes \mathbb{C}[[t]]$:

$$[x_1 \otimes p_1, x_2 \otimes p_2] = [x_1, x_2] \otimes p_1 p_2.$$

The resulting Lie algebra will be noted $\tilde{W}_1$. We propose to study the continuous cohomology of $\tilde{W}_1$, and the result of this section will be the answer in dimension 0, 1, 2 and 3. The author ignores the complete cohomology. The first step is the reduction of the Chevalley–Eilenberg complex involving all continuous cochains to the subcomplex consisting of cochains of weight 0 under the action of the Euler field $e_0 = z(\mathrm{d}/\mathrm{d}z)$. A well-known theorem states that the inclusion of this subcomplex induces a cohomology equivalence.

Let us recall the weight-0-subcomplex in the case of W_1:

$$C^1_{(0)}(W_1) = \mathbb{C}\epsilon_0; \qquad C^2_{(0)}(W_1) = \mathbb{C}\epsilon_{-1}\wedge\epsilon_1;$$
$$C^3_{(0)}(W_1) = \mathbb{C}\epsilon_{-1}\wedge\epsilon_0\wedge\epsilon_1; \qquad C^p_{(0)}(W_1) = 0 \quad \forall p \geq 4.$$

In the case of $\tilde{W}_1$, the subcomplex is unfortunately infinite-dimensional; for example in degree 1, all $\epsilon_0 \otimes \mathbb{C}[T]$ has weight 0 (T being the dual of t). We will drop the tensor sign between ϵ_i and T^r in the following. An additional problem arises from the fact that cochains $\epsilon_i p_1(T) \otimes \epsilon_j p_2(T)$ are not necessarily antisymmetric in (i, j), but only in $((i, 1), (j, 2))$.

So the low degree terms of the weight-0-subcomplex in the case of $\tilde{W}_1$ read as follows:

$$C^1_{(0)}(\tilde{W}_1) = \bigoplus_{r\geq 0}\mathbb{C}T^r\epsilon_0;$$

$$C^2_{(0)}(\tilde{W}_1) = \bigoplus_{r,s}\mathbb{C}T^r\epsilon_{-1}\wedge T^s\epsilon_1 \oplus \bigoplus_{r,s}\mathbb{C}T^r\epsilon_0\wedge T^s\epsilon_0;$$

$$C^3_{(0)}(\tilde{W}_1) = \bigoplus_{r,s,t}\mathbb{C}T^r\epsilon_{-1}\wedge T^s\epsilon_0\wedge T^t\epsilon_1 \oplus \bigoplus_{r,s,t}\mathbb{C}T^r\epsilon_{-1}\wedge T^s\epsilon_{-1}\wedge T^t\epsilon_2 \oplus \bigoplus_{r,s,t}\mathbb{C}T^r\epsilon_0 \wedge T^s\epsilon_0\wedge T^t\epsilon_0.$$

To calculate the cohomology, we will explicitly compute kernel and image of the Chevalley–Eilenberg coboundary d.

2.1. $H^1(\tilde{W}_1)$

Let G_1, G_2 be 2 elements of $\tilde{W}_1$, explicitly $G_i = \sum G_{s_i,t_i}t^{s_i}e_{t_i}$.
We calculate the coboundary of $\sum_r a_r T^r\epsilon_0$:

$$\mathrm{d}\left(\sum_r a_r T^r\epsilon_0\right)(G_1, G_2) = \sum a_r T^r(t^{s_1+s_2})\epsilon_0((t_2-t_1)e_{t_1+t_2})$$
$$= \sum_{s_1,s_2} a_{s_1+s+2}(-2G_{s_1,1}G_{s_2,-1} + 2G_{s_1,-1}G_{s_2,1}).$$

To compare, we calculate $(\sum a_{r_1,r_2}T^{r_1}\epsilon_1\wedge T^{r_2}\epsilon_{-1})(G_1, G_2)$ as follows:

$$\left(\sum a_{r_1,r_2}T^{r_1}\epsilon_1\wedge T^{r_2}\epsilon_{-1}\right)(G_1, G_2)$$
$$= \frac{1}{2}\sum a_{r_1,r_2}(T^{r_1}\epsilon_1\otimes T^{r_2}\epsilon_{-1} - T^{r_2}\epsilon_{-1}\otimes T^{r_1}\epsilon_1)(G_1, G_2)$$
$$= \frac{1}{2}\sum_{s_1,s_2} a_{s_1,s_2}(G_{s_1,1}G_{s_2,-1} - G_{s_1,-1}G_{s_2,1}).$$

We deduce

$$\mathrm{d}\left(\sum_r a_r T^r \epsilon_0\right) = -4 \sum b_{r_1,r_2} T^{r_1}\epsilon_1 \wedge T^{r_2}\epsilon_{-1}$$

with $b_{r_1,r_2} = a_{r_1+r_2}$.

So we have that the kernel of the Chevalley–Eilenberg coboundary d in degree 1 is 0, and in conclusion

$$H^1(\tilde{W}_1) = 0.$$

Remark. *The same result follows from the method of the Laplace operator which works well in degrees* 1 *and* 2.

2.2. $H^2(\tilde{W}_1)$

From the result of the preceding subsection, we deduce that the two-coboundaries $\sum_{r_1,r_2} a_{r_1,r_2}(T^{r_1}\epsilon_{-1}) \wedge (T^{r_2}\epsilon_1)$ are characterized by the fact that

$$a_{r_1,r_2} = b_{r_1+r_2}. \tag{1}$$

There are no two-coboundaries in $\oplus_{r,s}\mathbb{C}T^r\epsilon_0 \wedge T^s\epsilon_0$.

Let $G_1, G_2, G_3 \in \tilde{W}_1$, i.e. explicitly $G_i = \sum_{s_i,t_i} G_{s_i,t_i} t^{s_i} e_{t_i}$ for $i = 1, 2, 3$.

$$\begin{aligned}
&\mathrm{d}\left(\sum_{r_1,r_2} a_{r_1,r_2}(T^{r_1}\epsilon_{-1}) \wedge (T^{r_2}\epsilon_1)\right)(G_1, G_2, G_3)\\
&= \frac{1}{2}\sum_{r_j,s_i,t_i} a_{r_1,r_2} G_{s_1,t_1} G_{s_2,t_2} G_{s_3,t_3} \times \{T^{r_1}(t^{s_1+s_2})\epsilon_{-1}((t_2-t_1)e_{t_1+t_2})T^{r_2}(t^{s_3})\epsilon_1(e_{t_3})\\
&\quad -T^{r_1}(t^{s_1+s_3})\epsilon_{-1}((t_3-t_1)e_{t_1+t_3})T^{r_2}(t^{s_2})\epsilon_1(e_{t_2})\\
&\quad +T^{r_1}(t^{s_2+s_3})\epsilon_{-1}((t_3-t_2)e_{t_2+t_3})T^{r_2}(t^{s_1})\epsilon_1(e_{t_1})\\
&\quad -T^{r_1}(t^{s_1+s_2})\epsilon_1((t_2-t_1)e_{t_1+t_2})T^{r_2}(t^{s_3})\epsilon_{-1}(e_{t_3})\\
&\quad +T^{r_1}(t^{s_1+s_3})\epsilon_1((t_3-t_1)e_{t_1+t_3})T^{r_2}(t^{s_2})\epsilon_{-1}(e_{t_2})\\
&\quad - T^{r_1}(t^{s_2+s_3})\epsilon_1((t_3-t_2)e_{t_2+t_3})T^{r_2}(t^{s_1})\epsilon_{-1}(e_{t_1})\big\}
\end{aligned}$$

Here, we have six terms because we antisymmetrized the three terms from the coboundary. Now evaluating gives:

$$\cdots = \frac{3}{2}\sum_{s_1,s_2,s_3}\left\{(a_{s_2+s_3,s_1} - a_{s_1+s_3,s_2})G_{s_1,-1}G_{s_2,-1}G_{s_3,2} + \cdots\right\} \tag{2}$$

$$+\frac{1}{2}\sum_{s_1,s_2,s_3}\left\{(a_{s_1+s_2,s_3} - a_{s_1,s_2+s_3})G_{s_1,-1}G_{s_2,0}G_{s_3,1} + \cdots\right\} \tag{3}$$

Observe that (2) gives a (coboundary-)contribution to $\oplus_{r,s,t}\mathbb{C}T^r\epsilon_{-1} \wedge T^s\epsilon_{-1} \wedge T^t\epsilon_2$, wheras (3) gives a contribution to $\oplus_{r,s,t}\mathbb{C}T^r\epsilon_{-1} \wedge T^s\epsilon_0 \wedge T^t\epsilon_1$.

Let us show now that all two-cocycles from $\oplus_{r,s}\mathbb{C}T^r\epsilon_{-1} \wedge T^s\epsilon_1$ are two-coboundaries:

Lemma 1. *The cocycle condition* $a_{s_2+s_3,s_1} - a_{s_1+s_3,s_2} = 0$ *implies condition* 1.

Proof. Let $r_1, r_2, s_1, s_2 \in \mathbb{N}$ such that $r_1 + r_2 = s_1 + s_2$. We have to show that $a_{r_1,r_2} = a_{s_1,s_2}$-using the cocycle condition.

Let us suppose without loss of generality that $r_1 < s_1$.

Thus we have $r_2 > s_2$, i.e. $\exists p : r_2 = p + s_2$.

$$a_{r_1,r_2} = a_{r_1,p+s_2} = a_{r_1+p,s_2}$$

because of the cocycle condition. But

$$r_1 + p = r_1 + r_2 - s_2 = s_1.$$

This shows $a_{r_1,r_2} = a_{s_1,s_2}$. □

In order to state the result on $H^2(\tilde{W}_1)$, we have to consider the contribution from $\oplus_{r,s}\mathbb{C}T^r\epsilon_0 \wedge T^s\epsilon_0$. The same type of computation as above shows that the cocycle condition reads $a_{s_1+s_3,s_2} = a_{s_2,s_1+s_3}$. But a_{r_1,r_2} should be anti-symmetric in (r_1, r_2) because it comes from an element of $\oplus_{r,s}\mathbb{C}T^r\epsilon_0 \wedge T^s\epsilon_0$. Thus it is 0.

Corollary 1.

$$H^2(\tilde{W}_1) = 0.$$

Remark. *The method using the Laplacian still works in degree* 2 *and shows the same result. For degree* 3, *it would be too cumbersome*.

Our result is also consistent with Zusmanovich's result [12] on the second homology space of current Lie algebras, because $W_1/[W_1, W_1] = 0$, $H^2(W_1) = 0$ and $HC_1(\mathbb{C}[[t]]) = 0$.

2.3. $H^3(\tilde{W}_1)$

2.3.1. *Contribution from* $\oplus_{r,s,t}\mathbb{C}T^r\epsilon_{-1} \wedge T^s\epsilon_0 \wedge T^t\epsilon_1$

Let us deal first with the most important part, the contribution from $\oplus_{r,s,t}\mathbb{C}T^r\epsilon_{-1} \wedge T^s\epsilon_0 \wedge T^t\epsilon_1$.

Let $G_i \in \tilde{W}_1$, i.e. $G_i = \sum_{s_i,t_i} G_{s_i,t_i} t^{s_i} e_{t_i}$ for $i = 1, 2, 3, 4$.

$$
\begin{aligned}
&\mathrm{d}\left(\frac{1}{6}\sum_{r_j,s_i,t_i} a_{r_1,r_2,r_3}(T^{r_1}\epsilon_{-1})\wedge(T^{r_2}\epsilon_0)\wedge(T^{r_3}\epsilon_1)\right)(G_1,G_2,G_3,G_4)\\
&=\frac{1}{6}\sum_{r_j,s_i,t_i} a_{r_1,r_2,r_3}G_{s_1,t_1}G_{s_2,t_2}G_{s_3,t_3}G_{s_4,t_4}\\
&\quad\times\{T^{r_1}(t^{s_1+s_2})T^{r_2}(t^{s_3})T^{r_3}(t^{s_4})\epsilon_{-1}((t_2-t_1)e_{t_1+t_2})\epsilon_0(e_{t_3})\epsilon_1(e_{t_4})\\
&\quad+(-T^{r_1}(t^{s_1+s_3})T^{r_2}(t^{s_2})T^{r_3}(t^{s_4})\epsilon_{-1}((t_3-t_1)e_{t_1+t_3})\epsilon_0(e_{t_2})\epsilon_1(e_{t_4}))\\
&\quad+(+\mathbb{T}^{r_1}(t^{s_1+s_4})T^{r_2}(t^{s_2})T^{r_3}(t^{s_3})\epsilon_{-1}((t_4-t_1)e_{t_1+t_4})\epsilon_0(e_{t_2})\epsilon_1(e_{t_3}))\\
&\quad+(+T^{r_1}(t^{s_2+s_3})T^{r_2}(t^{s_1})T^{r_3}(t^{s_4})\epsilon_{-1}((t_3-t_2)e_{t_2+t_3})\epsilon_0(e_{t_1})\epsilon_1(e_{t_4}))\\
&\quad+(-T^{r_1}(t^{s_2+s_4})T^{r_2}(t^{s_1})T^{r_3}(t^{s_3})\epsilon_{-1}((t_4-t_2)e_{t_2+t_4})\epsilon_0(e_{t_1})\epsilon_1(e_{t_3}))\\
&\quad+(+T^{r_1}(t^{s_3+s_4})T^{r_2}(t^{s_1})T^{r_3}(t^{s_2})\epsilon_{-1}((t_4-t_3)e_{t_3+t_4})\epsilon_0(e_{t_1})\epsilon_1(e_{t_2})+\cdots\}
\end{aligned}
$$

Here, the dots at the end mean that the above term is to be repeated five times in order to anti-symmetrize it.

The coboundary gives six terms. Evaluating gives two additional terms to these six terms because for $\epsilon_1(e_{k+l})$, one has two possibilities: $k=0$, $l=1$ and $k=2$, $l=-1$. By interchanging k and l, one doubles the number of terms, giving 16. So, in total, we have six times 16 equals 96 terms in the sum.

Let us write down this sum with many ellipses:

$$
\begin{aligned}
&=\frac{1}{6}\sum_{s_1,s_2,s_3,s_4}\{(a_{s_1+s_2,s_3,s_4}-a_{s_1+s_3,s_2,s_4}-a_{s_1,s_3,s_2+s_4}+a_{s_1,s_2,s_3+s_4})\\
&\quad\times G_{s_1,-1}G_{s_2,0}G_{s_3,0}G_{s_4,1}+\cdots+2(a_{s_3,s_1+s_4,s_2}+a_{s_1,s_3+s_2,s_4}\\
&\quad-a_{s_3,s_1+s_2,s_4}-a_{s_1,s_4+s_3,s_2})G_{s_1,-1}G_{s_2,1}G_{s_3,-1}G_{s_4,1}+\cdots+3(-a_{s_4,s_3,s_1+s_2}\\
&\quad+a_{s_1,s_3,s_2+s_4})G_{s_1,-1}G_{s_2,2}G_{s_3,0}G_{s_4,-1}+\cdots\}
\end{aligned}
$$

Splitting -1 or 1 into $(0,-1)$, $(-1,0)$ or $(0,1)$, $(1,0)$ gives terms of the first kind, splitting 0 into $(1,-1)$, $(-1,1)$ gives terms of the second kind, weighted with a 2 because $(t_2-t_1)=(1-(-1))=2$, splitting 1 into $(2,-1)$, $(-1,2)$ gives the terms of the last kind, weighted with a 3.

This gives us three types of cocycle conditions:

$$a_{s_1+s_2,s_3,s_4}-a_{s_1+s_3,s_2,s_4}=a_{s_1,s_3,s_2+s_4}-a_{s_1,s_2,s_3+s_4} \tag{4}$$

$$a_{s_3,s_1+s_4,s_2}+a_{s_1,s_3+s_2,s_4}=a_{s_3,s_1+s_2,s_4}+a_{s_1,s_4+s_3,s_2} \tag{5}$$

$$a_{s_4,s_3,s_1+s_2}=a_{s_1,s_3,s_2+s_4} \tag{6}$$

Recalling term (3) and the contribution from $\oplus_{r,s}\mathbb{C}T^r\epsilon_0\wedge T^s\epsilon_0$, one sees that coboundaries are those satisfying

$$a_{r_1,r_2,r_3}=b_{r_1+r_2,r_3}-b_{r_1,r_2+r_3} \tag{7}$$

or

$$a_{r_1,r_2,r_3}=b_{r_1+r_3,r_2}-b_{r_2,r_1+r_3}. \tag{8}$$

Lemma 2. *The* a_{r_1,r_2,r_3} *satisfying* (4)–(6) *can be recontructed from* (7) *and* (8), *except* $a_{0,0,0}$.

Proof. Eq. (7) is 0 for $r_2 = 0$. Otherwise, one has three free parameters, so we can reconstruct a_{r_1,r_2,r_3}, except $a_{k,0,l}$.

One easily sees that (4)–(6) permit to construct all $a_{m,0,n}$ with fixed $m + n = k + l$ from a given $a_{k,0,l}$, so the cocycles which cannot be reconstructed in this step are the $a_{k,0,l}$ parametrized by the sum $k + l$. Eq. (8) permits to construct all a_{r_1,r_2,r_3}, except those with $r_1 = r_2$ in case $s_3 = 0$ or those with $r_2 = r_3$ in case $s_1 = 0$. The upshot of these considerations is that $a_{0,0,0}$ is the only cocycle-coefficient which cannot be recontructed using (7) and (8). □

So, the contribution from $\oplus_{r,s,t}\mathbb{C}T^r\epsilon_{-1} \wedge T^s\epsilon_0 \wedge T^t\epsilon_1$ to $H^3(\tilde{W}_1)$ is one-dimensional and generated by the image of the Godbillon–Vey cocycle under the map

$$H^3(W_1) \hookrightarrow H^3(\tilde{W}_1)$$

which is induced by the Lie algebra homomorphism

$$\tilde{W}_1 \to W_1, \qquad \sum_{r,s} a_{r,s}t^r e_s \mapsto \sum_s a_{0,s}e_s.$$

2.3.2. Contributions from $\oplus_{r,s,t}\mathbb{C}T^r\epsilon_{-1} \wedge T^s\epsilon_{-1} \wedge T^t\epsilon_2$ *and* $\oplus_{r,s,t}\mathbb{C}T^r\epsilon_0 \wedge T^s\epsilon_0 \wedge T^t\epsilon_0$ *are 0*

Now, let us show that the contributions from $\oplus_{r,s,t}\mathbb{C}T^r\epsilon_{-1} \wedge T^s\epsilon_{-1} \wedge T^t\epsilon_2$ and $\oplus_{r,s,t}\mathbb{C}T^r\epsilon_0 \wedge T^s\epsilon_0 \wedge T^t\epsilon_0$ are 0.

$$\begin{aligned} &\mathrm{d}\left(\frac{1}{6}\sum_{r_j,s_i,t_i} a_{r_1,r_2,r_3}(T^{r_1}\epsilon_0) \wedge (T^{r_2}\epsilon_0) \wedge (T^{r_3}\epsilon_0)\right)(G_1, G_2, G_3, G_4) \\ &= \frac{1}{3}\sum_{s_1,s_2,s_3,s_4}\{\mathrm{antisym}_{s_1+s_2,s_3,s_4} a_{s_1+s_2,s_3,s_4} G_{s_1,-1}G_{s_2,1}G_{s_3,0}G_{s_4,0} + \cdots\}\end{aligned}$$

Here, $\mathrm{antisym}_{k,l,m} a_{k,l,m}$ means the antisymmetrization of $a_{k,l,m}$ in the three indices. So the cocycle condition just means that $a_{r_1,r_2,r_3} = 0$ and the contribution from $\oplus_{r,s,t}\mathbb{C}T^r\epsilon_0 \wedge T^s\epsilon_0 \wedge T^t\epsilon_0$ to $H^3(\tilde{W}_1)$ is 0.

For the contribution from $\oplus_{r,s,t}\mathbb{C}T^r\epsilon_{-1} \wedge T^s\epsilon_{-1} \wedge T^t\epsilon_2$, we compute

$$\begin{aligned} &\mathrm{d}\left(\frac{1}{6}\sum_{r_j,s_i,t_i} a_{r_1,r_2,r_3}(T^{r_1}\epsilon_{-1}) \wedge (T^{r_2}\epsilon_{-1}) \wedge (T^{r_3}\epsilon_{-1})\right)(G_1, G_2, G_3, G_4) \\ &= \frac{1}{6}\sum_{s_1,s_2,s_3,s_4}\{(a_{s_1+s_2,s_3,s_4} + a_{s_1,s_2+s_3,s_4} - 2a_{s_1,s_3,s_2+s_4})G_{s_1,-1}G_{s_2,0}G_{s_3,-1}G_{s_4,2} \\ &\quad + \cdots + 4(a_{s_3,s_4,s_1+s_2} + a_{s_4,s_1,s_2+s_3} + a_{s_1,s_3,s_2+s_4})G_{s_1,-1}G_{s_2,3}G_{s_3,-1}G_{s_4,-1} + \cdots\}\end{aligned}$$

So, here are two types of cocycle conditions and the coboundary condition reads $a_{r_1,r_2,r_3} = b_{r_2+r_3,r_1} - b_{r_1+r_3,r_2}$.

Actually, this last condition expresses just the requirement for a_{r_1,r_2,r_3} to be antisymmetric with respect to (r_1, r_2). This is natural because it comes from an element in $\oplus_{r,s,t}\mathbb{C}T^r\epsilon_{-1}\wedge T^s\epsilon_{-1}\wedge T^t\epsilon_2$. We see that all elements can be written as coboundaries, making the contribution from $\oplus_{r,s,t}\mathbb{C}T^r\epsilon_{-1}\wedge T^s\epsilon_{-1}\wedge T^t\epsilon_2$ to $H^3(\tilde{W}_1)$ equally 0.

Now we can summarize the content of three subsections:

Theorem 1. *The Lie algebra homomorphism*

$$\phi : \tilde{W}_1 \to W_1, \qquad \sum_{r,s} a_{r,s}t^r e_s \mapsto \sum_s a_{0,s}e_s$$

induces a cohomology equivalence at least in degrees 0, 1, 2 *and* 3. *Explicitly*

$$\phi^p : H^p(W_1) \cong H^p(\tilde{W}_1) \quad \text{for} \quad p = 0, 1, 2, 3.$$

It is natural to conjecture that ϕ^p is an isomorphism for all p.

3. Second cohomology of *Vect*$_{1,0}(\Sigma)$

Let Σ be a compact Riemann surface of genus g. Let $T_p^{\mathbb{C}}|_{\mathrm{hol}}\Sigma$ denote the holomorphic part of the complexified tangent space $T_p^{\mathbb{C}}\Sigma$. Let *Vect*$_{1,0}(\Sigma)$ denote the space of C^∞ sections of the holomorphic vector bundle $T^{\mathbb{C}}|_{\mathrm{hol}}\Sigma := \bigcup_{p\in\Sigma}T_p^{\mathbb{C}}|_{\mathrm{hol}}\Sigma$. It is closed under the usual Lie bracket of vector field and thus a topological (Fréchet nuclear) Lie algebra. It is rather astonishing that the formal vector field Lie algebra which enters in the cohomology of *Vect*$_{1,0}(\Sigma)$ is not $\tilde{W}_1$, but W_1.

The main result of this section is

Theorem 2. *Let M be a complex manifold of complex dimension n.*

There is a spectral sequence for the diagonal cohomology of Vect$_{1,0}(M)$ *with second term*

$$E_2^{p,q} \cong H_{\partial}^{-p,0}(M)' \otimes H^q(W_n).$$

We deduce immediately the following

Corollary 2. *Let* Σ *be a compact Riemann surface of genus* $g > 1$.

There is a (*converging*) *spectral sequence for the diagonal cohomology of Vect*$_{1,0}(\Sigma)$ *with second term*

$$E_2^{p,q} \cong H_{\bar{\partial}}^{-p,0}(\Sigma)' \otimes H^q(W_1).$$

Some remarks are in order:

Remark 1. $E_2^{p,q}$ *does not involve* $H^q(\tilde{W}_1)$, *but* $H^q(W_1)$, *because — as we will see below — there is one real dimension missing. This is not important for* $H^2(\mathit{Vect}_{1,0}(\Sigma))$ *as we have* *an isomorphism* $H^q(W_1) \cong H^q(\tilde{W}_1)$ *for* $q \leq 3$. *If the conjecture stated in the last section is true, it does not even matter for general* $q \in \mathbb{N}$.

Remark 2. $H_{\partial}^{-p,0}(\Sigma)$ *denotes the* ∂-*cohomology* (*recall* $d = (\partial + \bar{\partial})$) *of the space of* C^∞ *differential forms of type* $(-p, 0)$, $H_{\bar{\partial}}^{-p,0}(\Sigma)$ *denotes its* $\bar{\partial}$-*cohomology. In particular, it is non-zero only for* $-p \geq 0$. $H_{\bar{\partial}}^{-p,0}(\Sigma)'$ *denotes its* (*continuous*) *dual.*

Concretely, we have

$$H_{\bar{\partial}}^{0,0}(\Sigma) = \mathbb{C}, \qquad H_{\bar{\partial}}^{1,0}(\Sigma) = \mathbb{C}^g, \qquad H_{\bar{\partial}}^{-p,0}(\Sigma) = 0 \quad \text{for} \quad -p \neq 0, 1$$

Remark 3. *By the dimensions of the cohomology spaces, it is obvious that*

$$\bigoplus_{p+q=1} E_2^{p,q} \cong 0 \quad \text{and} \quad \bigoplus_{p+q=2} E_2^{p,q} \cong \mathbb{C}^g.$$

This shows that $H_\Delta^1(\mathit{Vect}_{1,0}(\Sigma))$ *is* 0 *and* $H_\Delta^2(\mathit{Vect}_{1,0}(\Sigma))$ *is at most of dimension g. On the other hand, we have g independent generators, so that* $H_\Delta^2(\mathit{Vect}_{1,0}(\Sigma))$ *is exactly of dimension g. Explicitly, we have for genus* $g \geq 1$:

$$c\left(f(z,\bar{z})\frac{\partial}{\partial z}, g(z,\bar{z})\frac{\partial}{\partial z}\right) = \int_\Sigma \left\{ \begin{vmatrix} f & g \\ f'''_{z^3} & g'''_{z^3} \end{vmatrix} - 2R \begin{vmatrix} f & g \\ f'_z & g'_z \end{vmatrix} \right\} \mathrm{d}z \wedge \theta$$

Here, θ *is an anti-holomorphic* 1-*form, the intersection-dual of an element in* $H_{\bar{\partial}}^{1,0}(\Sigma)$. *R is a projective connection on* Σ *— this term is added in order to have globally defined holomorphic* 1-*form.*

Corollary 3. *Let* Σ *be a compact Riemann surface of genus* $g > 1$.

$$\dim(H_\Delta^2(\mathit{Vect}_{1,0}(\Sigma))) = g$$

As $H_\Delta^1(\mathit{Vect}_{1,0}(\Sigma)) = 0$, *we have a universal central extension of* $\mathit{Vect}_{1,0}(\Sigma)$ *given by the g generators, with center* $H_{\bar{\partial}}^{1,0}(\Sigma)'$.

Remark 4. *For genus* $g = 1$, *this corollary is already established as mentioned in* [2]. *Remark that* $\mathit{Vect}_{1,0}(\mathbb{T})$ *is just* $\Omega(\mathit{Vect}(S^1))$. *So, the result can also be easily deduced from* [12] *together with the Hochschild–Serre spectral sequence for the short exact sequence*

$$0 \to \Omega(\mathit{Vect}(S^1)) \to \Omega(\mathit{Vect}(S^1)) \times \left\langle \frac{\partial}{\partial z} \right\rangle \to \left\langle \frac{\partial}{\partial z} \right\rangle \to 0.$$

Remark 5. *Our theorem is consistent with Remark* 1 *in* [4, p. 76]: *There is a morphism of the dual of the Dolbeault complex into the cohomology complex of* $\mathit{Vect}_{1,0}(\Sigma)$; *for non-compact* Σ, $H^p(\mathit{Vect}_{1,0}(\Sigma))$ *is infinite dimensional for* $p = 3$, *for compact* Σ, *it is finite-dimensional for all p.*

Proof of the theorem. We will rely heavily on [6, Theorem 2.4.1a., p. 144] or on the original reference [7].

The idea is to calculate Gelfand–Fuks cohomology of vector field Lie algebras using the spectral sequence induced from the diagonal filtration. To compute the E_2-term of this spectral sequence, Gelfand and Fuks propose another spectral sequence, relying on the order filtration, see [6]. We change this filtration in the complex context to the filtration which concerns only the z-jet of the section, and not the whole jet:

Recall that a cochain $c \in C^q(\mathit{Vect}_{1,0}(M))$ can be regarded as a generalized section of a suitable vector bundle $\otimes^q(T^{\mathbb{C}}|_{\rm hol}M)$ on M^q, [6, pp. 142 and 143]. In particular, the subspace of cochains of diagonal filtration k, $C^q_k(\mathit{Vect}_{1,0}(M))$, can be regarded as the space of generalized sections concentrated on $M^q_k \subset M^q$ with

$$M^k_q = \{(x_1,\dots,x_k) \in M^k \mid \forall (i_1,\dots,i_{q+1}) \subset (1,\dots,k)\, \exists (i_j,i_r) : x_{i_j} = x_{i_r}\}.$$

Now remark that $\otimes^q(T^{\mathbb{C}}|_{\rm hol}M)$ is a holomorphic vector bundle on the complex manifold M^q. In particular, the notion of a trivial m-jet in z of a section of $\otimes^q(T^{\mathbb{C}}|_{\rm hol}M)$ (in a point $x \in M^q$) is independent on the choice of the local coordinate z. We restrict our setting now to the diagonal subcomplex $C^*_1(\mathit{Vect}_{1,0}(M))$.

Definition. We say that a generalized section $c \in C^q_1(\mathit{Vect}_{1,0}(M))$ has order $\le m$ if $c(s) \equiv 0$ for all sections $s \in \Gamma(\otimes^q(T^{\mathbb{C}}|_{\rm hol}M))$ such that s has a trivial m-jet in a neighbourhood of every point of $\Delta(M)$. Denote:

$$F^m C^q_1(\mathit{Vect}_{1,0}(M)) = \{c \in C^q_1(\mathit{Vect}_{1,0}(M)) \mid c \ \text{ has order } \ \le q-m\}.$$

It is easy to see that this gives a filtration on the diagonal complex. Indeed, $d(F^m C^q_1(\mathit{Vect}_{1,0}(M))) \subset F^m C^{q+1}_1(\mathit{Vect}_{1,0}(M))$, because the bracket in $\mathit{Vect}_{1,0}(M)$ involves only derivatives with respect to z, [7, cf. 2.10, p. 198]. In addition, it is exhaustive, because a section $s \in \Gamma(\otimes^q(T^{\mathbb{C}}|_{\rm hol}M))$ with trivial ∞-jet in z is zero due to the z-dependence coming from the transition functions in the holomorphic bundle $\otimes^q(T^{\mathbb{C}}|_{\rm hol}M)$.

In conclusion, we have the same situation as Gelfand and Fuks, but the anti-holomorphic half is missing. $E_0^{p,q}$ of the spectral sequence associated to this filtration is the quotient of diagonal cochains $C^{p+q}_\Delta(\mathit{Vect}_{1,0}(M))$ which are of order $\le q$ (i.e. vanishing on elements having trivial q-jets (in z)) factored by those of order $< q$.

Gelfand and Fuks translate elements of $E_0^{p,q}$ into generalized sections (suitable anti-symmetrized) of the bundle

$$\hat{\varepsilon}_0^{p,q} = \mathrm{Hom}\left(S^q \mathrm{norm}_{M^{p+q}}\Delta, \left(\otimes^{p+q} T^{\mathbb{C}}|_{\rm hol}M\right)|_\Delta\right).$$

Here, $\mathrm{norm}_{M^{p+q}}\Delta$ is the (holomorphic) normal bundle of the submanifold $\Delta(M) \subset M^{p+q}$ (it is the quotient bundle of the (holomorphic) tangent bundle of M^q by the (holomorphic) tangent bundle of $\Delta(M)$). $S^q\mathrm{norm}_{M^{p+q}}\Delta$ relates to the fact that exactly the q-jet (in z) is non-zero, jets of vector fields on M^{p+q} restricted to the diagonal, thus involving the normal bundle. The restriction to the diagonal stems from the definition of diagonal cochains being concentrated on the diagonal.

The anti-symmetrized version is denoted

$$\varepsilon_0^{p,q} = \mathrm{Alt}\left(S^q \mathrm{norm}_{M^{p+q}}\Delta, \left(\otimes^{p+q} T^{\mathbb{C}}|_{\rm hol}M\right)|_\Delta\right).$$

Now, we pass to considerations on the fibres of these bundles:
Let V denote the fibre of $T^{\mathbb{C}}|_{\text{hol}}M$. The fibre of $\hat{\varepsilon}_0^{p,q}$ is thus

$$\text{Hom}(S^q\{(V\oplus\cdots\oplus V)/V_\Delta\}, \underbrace{V\otimes\cdots\otimes V}_{p+q}).$$

Here, V_Δ stands for the image of the diagonal inclusion $V\to V\oplus\cdots\oplus V$. Because of the restriction of the order filtration of Gelfand–Fuks to jets involving just z, only the fibre of the holomorphic tangent space is showing up in the above formula.

Now, we can follow the proof of Theorem 2.4.1a. in [6, p. 144] word by word. We get (cf. p. 147)

$$E_1^{p,q}=\Omega^{-p,0}(M)'\otimes H^q(W_1).$$

But it is also clear that the differential $d_1^{p,q}$ will be the restriction to $\Omega^{-p,0}(M)$ of the one evidenced by Gelfand and Fuks, thus:

$$d_1^{p,q}=\partial\otimes id:\Omega^{-p,0}(M)'\otimes H^q(W_1)\to\Omega^{-(p+1),0}(M)'\otimes H^q(W_1).$$

This shows the theorem. □

Note that we can transpose all the content of the above lines to $Vect_{0,1}(\Sigma)$ simply by interchanging z and $\bar{z}$, so we have the same cohomological situation for $Vect_{0,1}(\Sigma)$.

Remark 6. *It is easily seen that the spectral sequence for the order filtration* (*calculating the cohomology of the diagonal subcomplex*) *collapses for Riemann surfaces, because there are only* 2 *non-zero cohomology spaces. Generalizing the Gelfand–Fuks spectral sequences for the terms of* (*diagonal*) *filtration k* (*cf.* [6, p. 142]), *we arrive at:*

$$^{(k)}E_2^{p,q}=H_\partial^{-p,0}(\Sigma^k,\Sigma_{k-1}^k)'\otimes\left[\bigoplus_{q_1+\cdots+q_k=q}H^{q_1}(W_1)\otimes\cdots\otimes H^{q_k}(W_1)\right]$$

This shows that there are no contributions from other spaces to $H^l(Vect_{1,0}(M))$ *for* $l=0,1,2,3$. *In general, we conjecture that the spectral sequence for the* (*diagonal*) *filtration also collapses* (*for Riemann surfaces*) *showing that — as in the case of* C^∞ *vector fields — the diagonal cohomology generates multiplicatively the continuous cohomology. Unfortunately, we were not able to calculate*

$$H_\partial^{-p,0}(\Sigma^k,\Sigma_{k-1}^k)$$

for general k.

Let us compare this result with the (*hyper*)*cohomology of* (*the sheaf of*) *holomorphic vector fields on* Σ, *cf.* [4,11]:

$$\mathbb{H}^*(\Sigma, C^*_{\text{cont}}(\text{Hol}))=S^*[\eta,\omega_1,\ldots,\omega_{b_1(\Sigma)},\theta].$$

Here, $C^*_{\text{cont}}(\text{Hol})$ *is the sheaf of continuous cochain complexes of the sheaf of holomorphic vector fields Hol,* $S^*[\ldots]$ *denotes the graded symmetric algebra* (*Hopf algebra*) *in some* *generators and* $b_1(\Sigma) = \dim(H^1(\Sigma))$. *The generators* η, ω_i *and* θ *are respectively of degrees* 1, 2 *and* 3.

So, if the conjecture is true, the continuous cohomology of $Vect_{1,0}(\Sigma)$ *constitutes a part of* $\mathbb{H}^*(\Sigma, C^*_{\mathrm{cont}}(\mathrm{Hol}))$, *namely half of the generators in degree* 2 *and the generator in degree* 3.

4. A two-dimensional analogue of Virasoro algebra

In this section we just recall known facts on the Virasoro algebra and construct by analogy the universal central extension of $Vect_{0,1}(\Sigma)$.

4.1. Virasoro algebra

The Virasoro algebra Vir is the universal central extension of $Vect(S^1)$ by means of the Gelfand–Fuks cocycle:

$$c\left(f(\theta)\frac{\mathrm{d}}{\mathrm{d}\theta}, g(\theta)\frac{\mathrm{d}}{\mathrm{d}\theta}\right) = \int_{S^1} \begin{vmatrix} f' & g' \\ f'' & g'' \end{vmatrix} (\theta)\,\mathrm{d}\theta$$

For a discussion of Vir from the point of view of universal central extensions, see [10].

A different characterization of Vir is the following: Let g be a (finite-dimensional) simple Lie algebra. Let $C^\infty(S^1, g)$ be the (Fréchet topological) vector space of C^∞ maps from S^1 to g. It is a Lie algebra by the pointwise bracket, dentoted Lg. Let $\hat{L}g$ be the central extension of Lg given by the Kac–Moody cocycle

$$\psi(f(\theta), g(\theta)) = \int_{S^1} \langle f, g'\rangle(\theta)\,\mathrm{d}\theta.$$

Here, $\langle , \rangle$ is the Killing form of g.

In this context, Vir is the universal central extension of the Lie algebra $\mathrm{Out}(\hat{L}g)$ of outer derivations of $\hat{L}g$. Let us briefly show this well-known fact.

Theorem 3.

$$\mathrm{Out}(\hat{L}g) = Vect(S^1).$$

Proof. As shown using Ex. 7.2–7.5 in [9, pp. 82 and 83], we have

$$\mathrm{Der}(Lg) = Vect(S^1) \oplus ad(Lg).$$

Here, $\mathrm{Der}(q)$ denotes the Lie algebra of derivations of the Lie algebra q and $ad(q)$ is its subspace of inner derivations.

Now, it is easy to see that the map $\phi : \mathrm{Der}(\hat{q}) \to \mathrm{Der}(q)$, $D \mapsto D$ is injective in case q is perfect — the kernel of ϕ being those derivations with values in the center. Therefore, we must look for derivations of $\hat{L}g$ inside $\mathrm{Der}(Lg)$. The condition for $D \in \mathrm{Der}(Lg)$ to be in $\mathrm{Der}(\hat{L}g)$ is (as is easily shown)

$$D(K) = \frac{\psi(D(f), g) + \psi(f, D(g))}{\psi(f, g)} K,$$

for all $f, g \in Lg$ — K being the central element ($\hat{L}g = Lg \oplus \mathbb{C}K$ as vector spaces).

This condition is satisfied for $D \in \mathit{Vect}(S^1)$ and $D \in ad(g) \subset ad(Lg)$ — the factor in front of K being 0. In conclusion:

$$\mathrm{Der}(\hat{Lg}) = \mathit{Vect}(S^1) \oplus ad(Lg) \quad \text{and} \quad \mathrm{Out}(\hat{L}g) = \mathit{Vect}(S^1). \qquad \square$$

4.2. Generalization

Here we follow closely [2].

Let Σ be a compact Riemann surface of genus g. Let g^Σ denote the current (Lie) algebra, i.e. the Lie algebra of C^∞ maps from Σ to g, g being a simple Lie algebra with its Killing form $\langle , \rangle$. By a well-known theorem of S. Bloch and B. Feigin, cf. [2], the universal central extension of g^Σ is an extension by the space $HC_1(C^\infty(\Sigma)) = \Omega^1(\Sigma)/\mathrm{d}\Omega^0(\Sigma)$, the quotient space of all 1-forms on Σ by the subspace of exact 1-forms. The $\Omega^1(\Sigma)$-valued cocycle defining this extension is

$$u(f, g) = \langle f, g\rangle \mod \mathrm{d}\Omega^0(\Sigma).$$

Etingof and Frenkel had the idea to restrict this extension to one with a finite-dimensional center by considering only the 1-forms compatible with a fixed complex structure on Σ, i.e. the holomorphic 1-forms.

Thus, let H_Σ be the space of holomorphic differentials on Σ. It is of dimension g. Let $\omega \in H_\Sigma \otimes H_\Sigma^*$ be the identity element, seen as a holomorphic differential on Σ with values in H_Σ^*. Define a 2-cocycle on g^Σ with values in the trivial g^Σ-module H_Σ^* by the formula

$$\Omega(f, g) = \int_\Sigma \omega \wedge \langle f, dg\rangle,$$

where $f, g \in g^\Sigma$. This cocycle defines a g-dimensional central extension of g^Σ, denoted $\hat{g}^\Sigma$.

Now, let us cite [2, Proposition 1.3]:

Proposition 1 ([2, Proposition 1.3]). *If $g > 1$, the Lie algebra of outer derivations of $\hat{g}^\Sigma$ coincides with the Lie algebra $\mathit{Vect}_{0,1}(\Sigma)$ of all complex valued vector fields on Σ of type $(0, 1)$, i.e. of the form $u(z, \bar{z})(\partial/\partial\bar{z})$ for any local complex coordinate z, u being a smooth function.*

If $g = 1$, the Lie algebra of outer derivations is $\langle\partial/\partial z\rangle \times \mathit{Vect}_{0,1}(\Sigma)$.

For the proof, we can remark that the proof of Theorem 3 shows that $\mathrm{Out}(g^\Sigma) = \mathit{Vect}(\Sigma)$. Afterwards, one follows [2].

As given in Remark 3, Section 3, we have g independent two-cocycles on $Vect_{0,1}(\Sigma)$ (where one has to use their anti-holomorphic version). Corollary 2 confirms Etingof, Frenkel, Khesin and Roger's conjecture that — as already verified for $g = 1$, cf. [2] — this gives the universal central extension of $Vect_{0,1}(\Sigma)$, thus giving a two-dimensional analogue of the Virasoro algebra as characterized by Theorem 3.

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